Lecture 8 - Sep 29

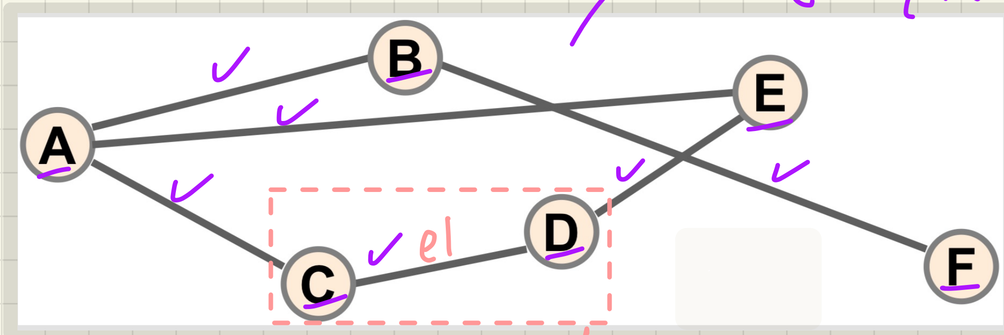
Graphs

Basic Definitions
Properties: Degrees, Number of Edges
Mathematical Induction on Vertices

Announcements/Reminders

- First Class (Syllabus) recording & notes posted
- Today's class: notes template posted
- Exercises:
  + Tutorial Week 1 (2D arrays)
  + Tutorial Week 2 (2D arrays, Proving Big-O)
  + Tutorial Week 3 (avg case analysis on doubling strategy)
  + Tutorial Week 4 (Trinode restructuring after deletions)

Graph: Definition



$V = \{A, B, C, D, E, F\}$

$E = \{(A, B), (A, C), (A, E), (C, D), (D, E), (B, F)\}$

$G = (V, E)$

vertices/nodes

edges/arcs

ordered pairs

$el = (C, D)$

$|V| = 6$

cardinality/size

$|E| = 6$

Edges: Directed vs. Undirected

peter — mary
markus

Directed

dependency, flow, order among nodes

$u \in V$
$v \in V$

$u \rightarrow v$

u: origin
v: destination

(u, v)
(v, u) undefined.

$(u, v) \neq (v, u)$

Undirected

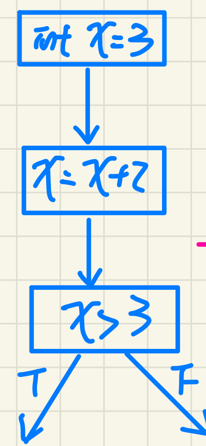
$u - v$

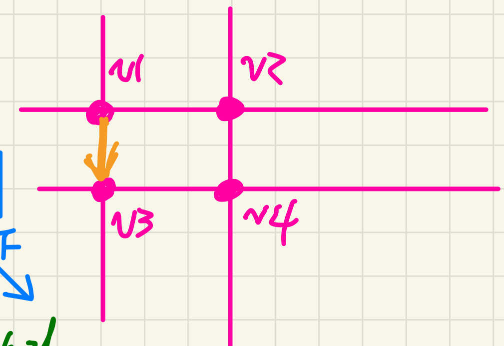
⇕

as if: (u, v) and (v, u) bi-directional.

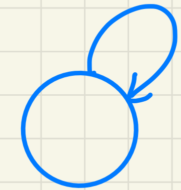
Examples:

- Control Flow/Data Flow Diagrams
- Social Network of Friendships (symmetry)
- Road Map of GPS → there's some one-way street
- Collaboration Network (Co-authorship)
- Degree Requirement pre-requisite. → topological sort.
- Web Pages (Hyperlinked)
- Protein-Protein Interaction Network → symmetry.



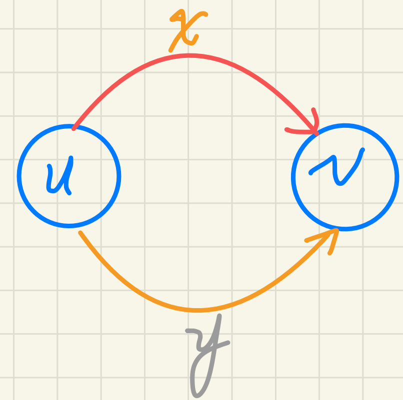


self edge/loop : (u, u)



multiple/parallel edges: (u, v)

(u, v)



Simple Graph : graph without self and parallel edges

not simple graph : graph has self edges or parallel edges.

Vertices: Degree

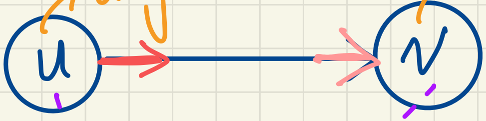
→ # of edges incident on a vertex

undirected: degree of vertex

directed: in-degree vs out-degree.

origin

destination

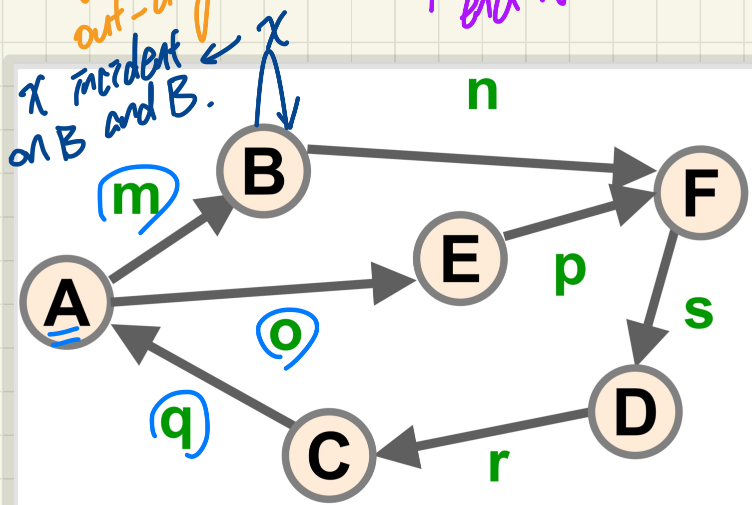


endpoints, end vertices

(u, v) is incident on vertices u and v.

- is outgoing edge of u
- is incoming edge of v

x incident on B and B.



Exercises:

End vertices of m? A, B

Outgoing Edges of A? m, o

Incoming Edges of A? q

Edges incident on A? m, o, q.

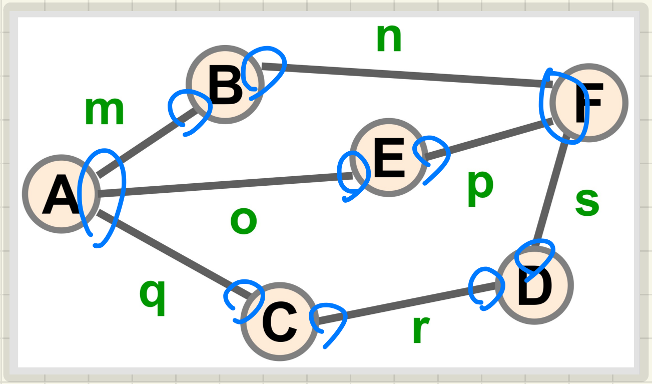
Degree of A? 3 (in-degree(A) = 1, out-degree(A) = 2)

Properties: Sum of Degrees for Undirected Graphs

Given a **simple**, **undirected** graph $G = (V, E)$ with $|E| = m$:

$$\sum_{v \in V} \text{degree}(v) = 2 \cdot m$$

$\hookrightarrow |E|$



Vertex	Degree
A	3
B	2
C	2
D	2
E	2
F	3

$|E| = 7$

m

$\boxed{14} = 2 \cdot |E|$

Properties: Sum of Degrees for Undirected Graphs

Given a (non-empty) **simple**, **undirected** graph $G = (V, E)$ with $|E| = m$:

$$\boxed{\sum_{v \in V} \text{degree}(v) = 2 \cdot m}$$ claim

Strategy of Proof : Perform a M.I. on $|V|$

(1) Base Case : $|V| = 1$

x $\quad |E| = 0.$

$\text{degree}(x) = 0 \qquad \sum_{v \in \{x\}} \text{degree}(v) = \text{degree}(x) = 0 = 2 \cdot \underbrace{|E|}_{0}$

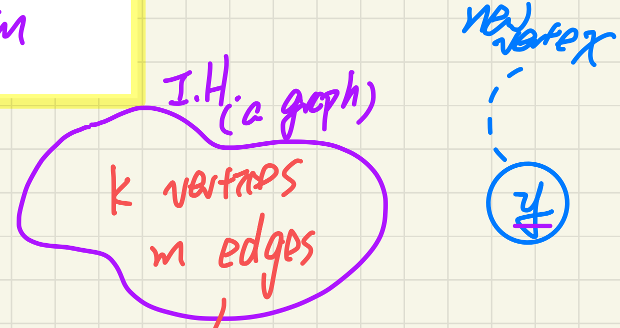
I.H. holds

(2) Inductive Hypothesis (I.H.)

$$\sum_{v \in V} \text{degree}(v) = 2 \cdot m$$

where $|V| = k > 1$

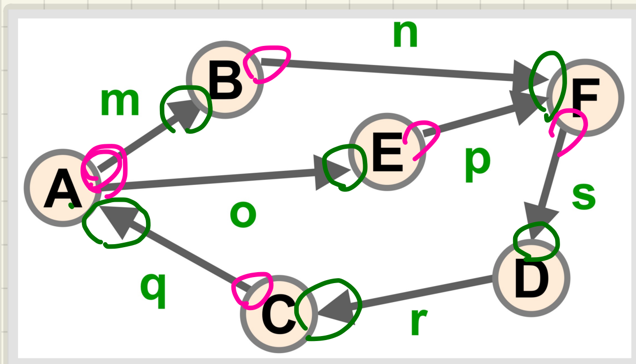
(3)* Make a strictly larger graph with $k+1$ vertices (by adding a new vertex y)



Properties: Sum of Degrees for Directed Graphs

Given a **simple**, **directed** graph $G = (V, E)$ with $|E| = m$:

$$\sum_{v \in V} \texttt{in-degree}(v) = \sum_{v \in V} \texttt{out-degree}(v)$$



Vertex	in-degree	out-degree
A	1	2
B	1	1
C	1	1
D	1	1
E	1	1
F	2	1
	$\Sigma = 7$	$\Sigma = 7 = \overset{m}{=} \lvert E \rvert$

Properties: Sum of Degrees for Directed Graphs

Given a **simple**, **directed** graph $G = (V, E)$ with $|E| = m$:

$$\sum_{v \in V} \texttt{in-degree}(v) = \sum_{v \in V} \texttt{out-degree}(v)$$

Properties: Sum of Degrees for Directed Graphs

Given a **simple**, **undirected** graph $G = (V, E)$, $|V| = n$, $|E| = m$:

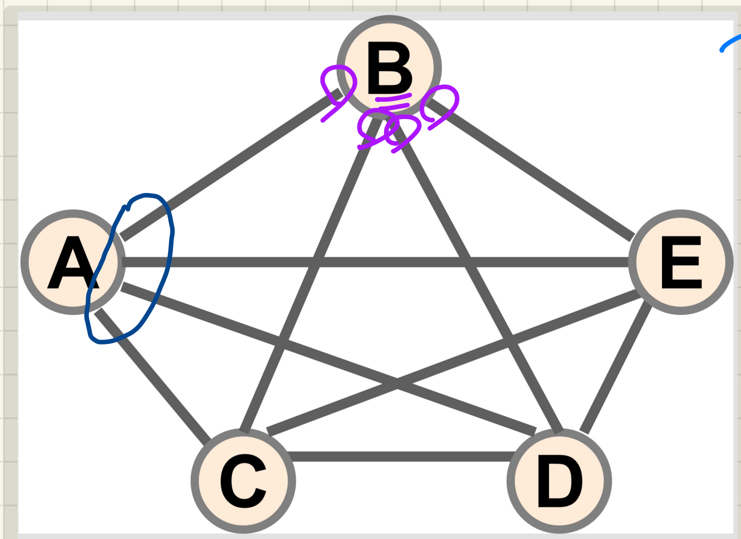
$$m \le \frac{n \cdot (n-1)}{2}$$

→ # of edges is $O(|V|^2)$

$|E|$

→ $|V| = 5$

$|E| = 10$



$$\frac{|V| \times (|V|-1)}{2}$$

Vertex	edges
A	(A, B), (A, C), (A, D), (A, E)
B	(B, A), (B, C), (B, D), (B, E)
C	
D	
E	

$|V|$

max if a vertex is connected to all other $|V|-1$ vertices

$|V|-1$

2 edges should be counted as 1 edge towards m

Given a **simple**, **undirected** graph $G = (V, E)$, $|V| = n$, $|E| = m$:

$$m \leq \frac{n \cdot (n-1)}{2}$$

$=$

When $m = \dfrac{n \cdot (n-1)}{2}$

$\Rightarrow$ G is Complete

Properties: Sum of Degrees for Directed Graphs

Given a **simple**, **undirected** graph $G = (V, E)$, $|V| = n$, $|E| = m$:

$$m \leq \frac{n \cdot (n-1)}{2}$$